2.5 The Quantum Eraser

The two slit experiment is often the first thought experiment a student encounters when studying quantum mechanics. Here we will explore some variants to it that highlight the curious interplay between coherence, interference and entanglement.

Standard two slit experiment (1) : Let us start with the standard two slit experiment. We suppose that single horizontally polarized photons impinge on a screen with two slits and hit a second screen placed behind the first (see Fig. 2.1a)). Although the photons hit the screen one by one we see an interference pattern on the screen behind.

Standard two slit experiment (2) : We now suppose that a 90 degrees polarisation shifter is placed behind one of the slits (so that the light coming through it now is vertically polarized) but otherwise leave the set up unchanged (Fig. 2.1b). *What happens this time ?*

In this case the interference pattern does not arise. Instead we see a simple mixture of the two patterns we would get if the photons went either through the top or the bottom slit as shown in Fig. 2.1b. This is because if we measured each photons polarisation then we would be able to determine if it went through the top or the bottom slit. Even if we do not in fact check which slit we went through this information is enough to destroy the interference pattern.

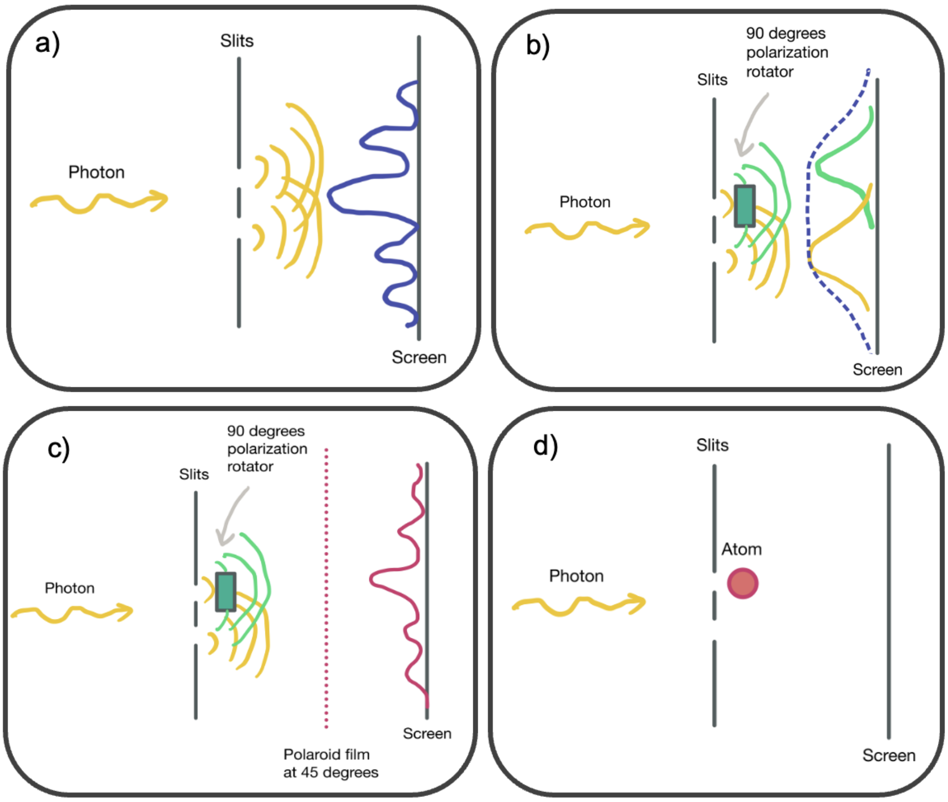

FIGURE 2.1 – Quantum eraser experiments

Here is how to understand this mathematically. Let $\psi_1(x, t)$ be the wavefunction of a photon emerging from the first slit, and $\psi_2(x,t)$ be that from the second slit. Let the polarisation of a photon be labelled by a H (horizontal) or V (vertical) substate, so that a horizontally-polarised photon emerging from the first slit is written as $|\psi_1, H\rangle = |\psi_1\rangle \otimes |H\rangle$. In the original two slit experiment the state of the photon after going through the two slits is of the form

$$|\Psi(x,t)\rangle = \frac{1}{\sqrt{2}}(|\psi_1(x,t)\rangle + |\psi_2(x,t)\rangle) \otimes |H\rangle \tag{2.2}$$

and on measuring the position of the particle at the second screen we get the probability density

$$\begin{aligned} P(x) &= \langle\Psi(x,t)|\,(|x\rangle\langle x| \otimes I)\,|\Psi(x,t)\rangle \\ &= \frac{1}{2}(\langle\psi_1(x,t)| + \langle\psi_2(x,t)|)|x\rangle\langle x|(|\psi_1(x,t)\rangle + |\psi_2(x,t)\rangle)\langle H|H\rangle \\ &= \frac{1}{2}\,\langle\psi_1(x,t) + \psi_2(x,t)|\,x\rangle\langle x\,|\psi_1(x,t) + \psi_2(x,t)\rangle \\ &= |\psi_1(x,t) + \psi_2(x,t)|^2/2\,. \end{aligned} \tag{2.3}$$

In the second case the state of the photon after passing through the two slits and the polarization shifter is of the form

$$|\Phi(x,t)\rangle = \frac{1}{\sqrt{2}}(|\psi_1(x,t)\rangle \otimes |V\rangle + |\psi_2(x,t)\rangle \otimes |H\rangle) \tag{2.4}$$

and so the probability density function of the photons hitting the screen is

$$\begin{aligned} P(x) &= \langle\Phi(x,t)|\,(|x\rangle\langle x| \otimes I)\,|\Phi(x,t)\rangle \\ &= \langle\psi_1(x,t)|\,x\rangle\langle x\,|\psi_1(x,t)\rangle\,\langle V|V\rangle + \langle\psi_2(x,t)|\,x\rangle\langle x\,|\psi_2(x,t)\rangle\,\langle H|H\rangle \\ &= (|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2)/2 \end{aligned} \tag{2.5}$$

That is we have a probabilistic mixture because the cross terms, the interference terms, have vanished because $\langle H|V\rangle = 0$.

Quantum eraser : We now suppose that as well as the 90 degrees polarisation shifter behind one of the slits we add a polaroid sheet at 45 degrees, which only outputs light in the state $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$. This is shown in Fig. 2.1c). *What happens this time ?*

We see the interference pattern again but at half the intensity. Why ? The light coming through the top slit is vertically polarized and the photons coming through the bottom slit is horizontally polarized. The polaroid sheet effectively measures the polarization degree of freedom in the $\{|\nearrow\rangle, |\swarrow\rangle\}$ basis where $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and $|\swarrow\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$, and only lets through measurement outcomes that project the light to $|\nearrow\rangle$. Now both H and V photons have a 50% chance of being measured to be $|\nearrow\rangle$ and so the sheet lets through only half the photons. But crucially all the photons (both the ones from the upper slit and the lower slit) that get let through are in the $|\nearrow\rangle$ state and so it's impossible to determine which slit any photon went through.

Lets see how this looks mathematically. First let's rewrite the Φ state in the $\{|\nearrow\rangle, |\swarrow\rangle\}$ basis,

$$\begin{aligned} |\Phi(x,t)\rangle &= \frac{1}{\sqrt{2}}(|\psi_1(x,t)\rangle \otimes |V\rangle + |\psi_2(x,t)\rangle \otimes |H\rangle) \\ &= \frac{1}{2}(|\psi_1(x,t)\rangle \otimes (|\nearrow\rangle - |\swarrow\rangle) + |\psi_2(x,t)\rangle \otimes (|\nearrow\rangle + |\swarrow\rangle)) \end{aligned} \tag{2.6}$$

After going through the filter the state becomes

$$|\Phi(x,t)'\rangle = \frac{1}{2}(|\psi_1(x,t)\rangle \otimes |\nearrow\rangle + |\psi_2(x,t)\rangle \otimes |\nearrow\rangle) \tag{2.7}$$

This is of the same form as Eq. (2.2) except i. we have an extra factor of $1/\sqrt{2}$ out the front and ii. all the photons are now in the $\nearrow$ polarization state instead of the H state. It follows that the interference pattern is the same as Eq. (2.3) but with an extra factor of 1/2 out the front. That is we see the interference pattern but with the intensity reduced by 1/2 as claimed :

$$P(x) = |\psi_1(x,t) + \psi_2(x,t)|^2/4\,. \tag{2.8}$$

Exercise : What changes if the polaroid sheet only lets through $|\swarrow\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$ photons ?

Delayed quantum eraser : Let's go back to the simple two slit experiment and this time place an atom behind one of the slits as sketched in Fig. 2.1d). Now this would be hard to arrange in practise but let us suppose that the photon that passes the atom flips the spin of an outer electron from $|\downarrow\rangle$ to $|\uparrow\rangle$ but is not absorbed[2]. (For each photon that we send through the two slit experiment we use a new atom and store the previous in a quantum memory). What happens in this case ?

Concretely, after passing through the two slits and past the atom the system is in the state :

$$\begin{aligned} |\Phi(x,t)\rangle &= \frac{1}{\sqrt{2}}(|\psi_1(x,t)\rangle \otimes |\uparrow\rangle + |\psi_2(x,t)\rangle \otimes |\downarrow\rangle) \\ &= \frac{1}{2}(|\psi_1(x,t)\rangle \otimes (|\nearrow\rangle - |\swarrow\rangle) + |\psi_2(x,t)\rangle \otimes (|\nearrow\rangle + |\swarrow\rangle)) \end{aligned} \tag{2.9}$$

Then we can read off the expected interference patterns in the different cases :

— **Measure in the Z basis :**

If we obtain $|\uparrow\rangle$ then the pattern is $|\psi_1(x,t)|^2$.
If we obtain $|\downarrow\rangle$ then the pattern is $|\psi_2(x,t)|^2$.

— **Measure in the X basis :**

If we obtain $|\nearrow\rangle$ then the interference pattern is $\frac{1}{2}|\psi_1(x,t) + \psi_2(x,t)|^2$.
If we obtain $|\swarrow\rangle$ then the interference pattern is $\frac{1}{2}|\psi_1(x,t) - \psi_2(x,t)|^2$.

So it would seem that the interference pattern we observe depends on the basis that the atom is measured in. If the atom is measured in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ then we end up with version 2 of the standard two slit experiment where we know which slit the photon went through. However, if we measure in the $\{|\nearrow\rangle, |\swarrow\rangle\}$ basis we end up with the quantum eraser version, and the interference reappears (but we do not lose half the photons this time).

The interference pattern depends on the basis that the atom is measured in - something we subjectively choose. And more puzzling still, this is true even if the atoms are taken far away before being measured ! So a natural thought might be - can we use this to signal ?

[Some blank space to encourage you to think about this before reading the answer]

2. An experiment of this spirit but not of this exact form has been conducted. Take a look at the wikipedia page on the delayed quantum eraser to learn more.

2.6 No signalling

Ok, so could we use the delayed eraser setup for a superluminal signal ? On the surface it might look like we should be able to. Suppose Bob can perform measurements on the atom, and Alice watches the screen subsequently impacted by photons. They try and signal (Bob is the sender, Alice is the receiver) using the code that an interference pattern corresponds to the bit '0' and no interference corresponds to the bit '1'. Then, it would seem that Bob could measure Z or X to send '0' or '1' to Alice and this would be true no matter how far away he is from Alice, seemingly allowing superluminal signalling. However, if Bob could signal to Alice in this way it would violate special relativity. So what breaks down ?

Well the key thing to note is that the interference pattern depends on not just the measurement, but the measurement *outcome*. Say the atom is measured in the Z basis. Bob will obtain $|\uparrow\rangle$ and $|\downarrow\rangle$ with equal probabilities (because the photon is equally likely to go through either slits) and so the resulting pattern on the screen is

$$p(x) = (|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2)/2\,. \tag{2.10}$$

Similarly, if Bob measures in the X basis then the states $|+\rangle$ and $|-\rangle$ are obtained with equal probabilities and so the resulting pattern is

$$p(x) = (|\psi_1(x,t) + \psi_2(x,t)|^2 + |\psi_1(x,t) - \psi_2(x,t)|^2)/2 = (|\psi_1(x,t)|^2 + |\psi_2(x,t)|^2)/2\,. \tag{2.11}$$

That is, the pattern is the same in either case !

In order to be able to communicate with this set up Bob would need to tell Alice for each photon that went through the setup which outcome he obtained. She could then mark the photons according to the outcome obtained and determine whether or not an interference pattern was observed for measurement outcomes of the same sort (corresponding to X measurement) or no interference pattern (corresponding to Z measurement). However, this requires communication which defeats the purpose of the purported signalling protocol.

Ok, so this quantum erasor protocol doesn't work. Could another more general protocol work ? Suppose Alice and Bob each have a qubit of a generic entangled state $|\Psi\rangle$ that they want to use to try and signal. Suppose Bob considers performing two different measurements upon his qubit ; $M^{(B1)}$ which has outcomes corresponding to projectors $\Pi_0^{(B1)}$ and $\Pi_1^{(B1)}$, and $M^{(B2)}$ which corresponds to projectors $\Pi_0^{(B2)}$ and $\Pi_1^{(B2)}$. In words : the superscript indicates which measurement axis he chose, and the subscript indicates what outcome he obtained therefrom. Bob intends to signal a bit '0' or '1' to Alice via his choice of measurement. Suppose these measurements collapse Alice's state as follows :

1. **Bob measures $M^{(B1)}$, obtaining the outcome described by $\Pi_i^{(B1)}$:**
   Alice's qubit enters state $|\psi_i\rangle$ with probability p_i.
2. **Bob measures $M^{(B2)}$, obtaining the outcome described by $\Pi_i^{(B2)}$:**
   Alice's qubit enters state $|\phi_i\rangle$ with probability q_i.

Then in order for Alice to infer whether Bob measured $M^{(B1)}$ or $M^{(B2)}$, she must perform some measurement $M^{(A)}$ that, at the very least[3] , has different outcome probabilities depending on

3. What else would be required ? How can Alice determine the probabilities of her measurement outcomes ?

Bob's measurement. Let $\Pi^{(A)}$ be the projector of one of her possible measurement outcomes (it's not necessary to think about the other outcome). Alice requires that

$$P(\Pi^{(A)}|M^{(B1)}) \neq P(\Pi^{(A)}|M^{(B2)}) \tag{2.12}$$

$$\implies \sum_i p_i \langle\psi_i|\,\Pi^{(A)}\,|\psi_i\rangle \neq \sum_i q_i \langle\phi_i|\,\Pi^{(A)}\,|\phi_i\rangle\,. \tag{2.13}$$

It turns out that it is impossible to find such an operator. That is, for any choice of $\Pi^{(A)}$, the above expression is a strict *equality*. It follows that it is impossible to use an entangled state to communicate faster than the speed of light. For an example of this see this chapter's problem sheet. We will also demonstrate this more rigorously when we cover reduced states in a few lectures time.

2.7 Non-locality and Bell inequalities

In this section we will explore how quantum entanglement can produce correlations that cannot be explained by classical observers that pre-share classical correlated data/randomness. More concretely, we will see how Bell's theorem, and experimental verifications of it, imply that not only quantum physics but also our world is inherently 'non-local'. I will start this section with an unconventional way of framing the Bell's Theorem that I have shamelessly borrowed from Terry Rudolph.

2.7.1 Quantum Psychics

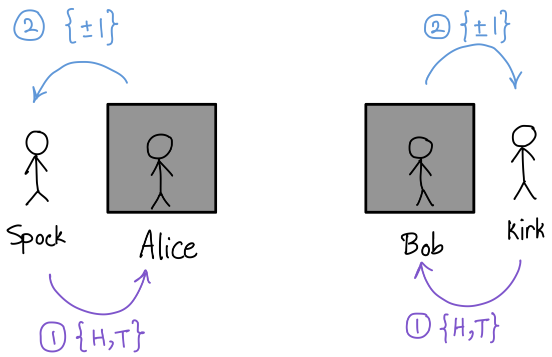

FIGURE 2.2 – **The Quantum Psychics Game.**

Suppose there were two friends Alice and Bob who claimed to share a psychic connection. How could you go about testing it ? Let's put Alice and Bob into isolated rooms with no way they can pass any messages between them. Outside Alice's room is a sceptic, let's call him Spock, who tosses a coin and tells Alice the outcome. Outside Bob's room is another sceptic, Kirk, who similarly tosses a coin and tells Bob the outcome. Alice and Bob must then respond with either yes 'Y' or no 'N'. What can Spock and Kirk ask Alice and Bob to do to try to determine if they are psychic ? They consider the following tests...

Test 1 : *Every time Alice and Bob get told the same coin outcome they must give the same answer, every time they get different outcomes they must give different answers.*

This clearly is a flawed test. Alice and Bob can pass it simply by deciding in advance that they will both say yes to heads and no to tails.

Realising this, the Spock and Kirk instead toy with proposing an alternative test...

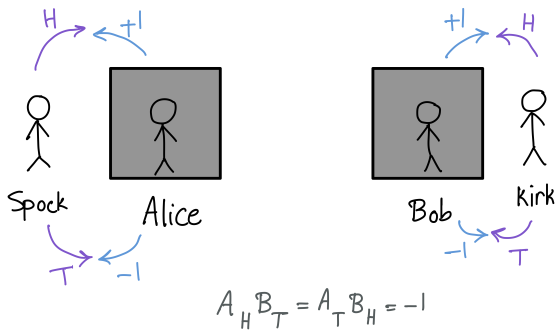

FIGURE 2.3 – **The Quantum Psychics Game : Test 1.**

Test 2 : *Every time Alice and Bob get told the same coin outcome they must give opposite answers, every time they get different flips they must give the same answers.*

On further thought this test is equally flawed. Alice and Bob agree in advance that they will give different outcomes. That is, Alice says yes to heads and no to tails but Bob does the converse.

Instead the Spock and Kirk propose the following test.

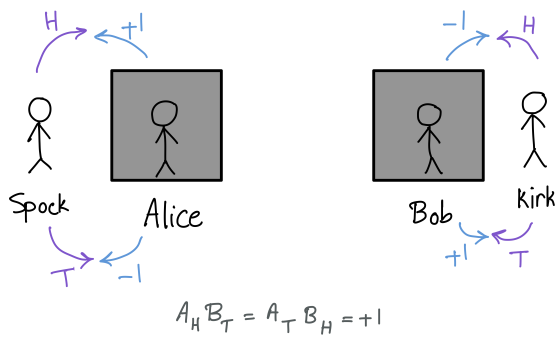

FIGURE 2.4 – **The Quantum Psychics Game : Test 2.**

Test 3 : *Every time Alice and Bob get told 'H' they must give opposite answers, but otherwise they must give the same answer.*

Now if you play around with this you'll see that there is no strategy that Alice and Bob can cook up in advance in order to fool the sceptics. Try this ! After playing with a few examples, the easiest way to definitively prove it to yourself is to represent the binary answers 'Y' and 'N' by +1 and -1 respectively. Then the rules of the game can be formalized as trying to find an assignment of A_H, A_T, B_H and B_T such that

$$\begin{aligned} A_H B_H &= -1 \\ A_H B_T &= 1 \\ A_T B_H &= 1 \\ A_T B_T &= 1 \end{aligned} \tag{2.14}$$

Multiplying the left hand side of these four equations together gives $A_H^2 A_T^2 B_H^2 B_T^2$ which has to be positive. However, multiplying the right hand side together gives -1. Hence there cannot be an assignment of A_H and B_H that satisfies all the rules of the test and as such this test is a viable means to testing if Alice and Bob are psychic.

In fact, the maximum number of rules that can be satisfied in Eq. 2.14 for any strategy taken by Alice and Bob is 3. (Convince yourself of this !) That is, at best Alice and Bob can pick a strategy that will lead to them fooling the sceptics for 3 out of the 4 possible coin toss combinations :

$$P_{\text{win}} \leq 3/4\,. \tag{2.15}$$

This is an example of a Bell inequality. If Alice and Bob reliably can win with a probability significantly greater than 3/4 then it would seem reasonable to assume that they really are 'psychic' (by which I mean, there are correlations that cannot be explained by previously decided classical scheme for correlating their answers).

However, if Alice and Bob share entangled Bell states, $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then they can use the non-classical correlations stored in the Bell state to pass the sceptics test. Alice and Bob's strategy to do so is as follows.

— If Alice gets told 'H' she measures in the Z basis and says 'Y' if she gets '$|0\rangle$' and 'N' if she gets '$|1\rangle$'.
— If Alice gets told 'T' she measures in the X basis and says 'Y' if she gets '$|+\rangle$' and 'N' if she gets '$|-\rangle$'.
— If Bob gets told 'H' he measures in the basis

$$\left\{|h\rangle = \sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle\,,\, |\bar{h}\rangle = \cos(\pi/8)|0\rangle - \sin(\pi/8)|1\rangle\right\} \tag{2.16}$$

and says 'Y' if he gets '$|h\rangle$' and 'N' if she gets '$|\bar{h}\rangle$'.
— If Bob gets told 'T' he measures in the basis

$$\left\{|t\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle\,,\, |\bar{t}\rangle = \sin(\pi/8)|0\rangle - \cos(\pi/8)|1\rangle\right\} \tag{2.17}$$

and says 'Y' if he gets '$|t\rangle$' and 'N' if she gets '$|\bar{t}\rangle$'.

Alice and Bob can beat test 3 with probability

$$P_{\text{Quantum}} = \cos(\pi/8)^2 = \frac{2+\sqrt{2}}{4} \approx 0.854\,. \tag{2.18}$$

Exercise : Check this !

However, crucially this is an intriguing form of telepathy. They can use it to cheat the sceptics test but (as we saw before and you will see in the problem sheet) they cannot use it to signal. So is it useful for anything ? In fact, it proves useful in quantum cryptography (but that is beyond the remit of this course).

Terry's quantum psychics version of the Bell inequality is entirely equivalent to a more conventional framing of the Bell's theorem known as the CHSH inequality. Rather than asking what is the probability of Alice and Bob winning test 3, the CHSH inequality is a bound on the sum of the expectation values of the product of Alice and Bob's answers for each of the different possible combinations of outcomes. That is, a bound on the correlation coefficient

$$C := \langle A_T B_T \rangle + \langle A_H B_T \rangle + \langle A_T B_H \rangle - \langle A_H B_H \rangle \tag{2.19}$$

where $A_j B_k$ are placeholders for Alice and Bob's measurement outcomes when told the toss outcome was j and k respectively. For example, A_H and B_H are placeholders when they are both told H and so

$$\begin{aligned} \langle A_H B_H \rangle = (-1) \times P(A_H = 1, B_H = -1 | H, H) + (-1) \times P(A_H = -1, B_H = 1 | H, H) \\ + (+1) \times P(A_H = 1, B_H = 1 | H, H) + (+1) \times P(A_H = -1, B_H = -1 | H, H) \,. \end{aligned} \tag{2.20}$$

and similarly for the other expectations values. We want to relate this to probability of winning in test 3,

$$\begin{aligned} P_{\text{win}} = \frac{1}{4} \Big(& P(A_H = 1, B_H = -1 | H, H) + P(A_H = -1, B_H = 1 | H, H) \\ & P(A_H = 1, B_T = 1 | H, T) + P(A_H = -1, B_T = -1 | H, T) \\ & P(A_T = 1, B_H = 1 | T, H) + P(A_T = -1, B_H = -1 | T, H) \\ & P(A_T = 1, B_T = 1 | T, T) + P(A_T = -1, B_T = -1 | T, T) \Big) \end{aligned} \tag{2.21}$$

To do so, we note that as the probability of the different outcomes have to sum to 1, we can write $\langle A_H B_H \rangle$ as

$$\langle A_H B_H \rangle = 1 - 2(P(A_H = 1, B_H = -1 | H, H) + P(A_H = -1, B_H = 1 | H, H)) \,. \tag{2.22}$$

On using a similar trick with the other expectations values, the probability of winning in test 3 is given by

$$P_{\text{win}} = \frac{1}{8} \left((1 - \langle A_H B_H \rangle) + (1 + \langle A_H B_T \rangle) + (1 + \langle A_T B_H \rangle) + (1 + \langle A_T B_T \rangle) \right) = \frac{1}{2} + \frac{1}{8} C \,. \tag{2.23}$$

As $P_{\text{win}} \leq 3/4$, it follows that

$$C = 8 \left(P_{\text{win}} - \frac{1}{2} \right) \leq 2 \,. \tag{2.24}$$

However for quantum players we have $P_{\text{quantum}} = \frac{1}{2} + \frac{\sqrt{2}}{4}$ and so

$$C_{\text{quantum}} = 2\sqrt{2} \,. \tag{2.25}$$

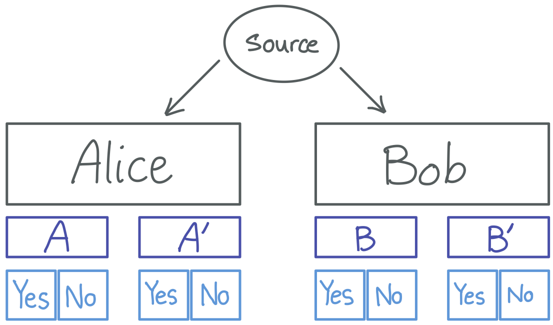

FIGURE 2.5 – **The CHSH Inequality**

2.7.2 More formal derivation (i.e. pinning down exactly what is spooky)

We introduced Bell inequalities above with a thought experiment about testing psychics. This hopefully helped to give you an intuition about what is so strange about violating a Bell inequality. Below we present a more formal derivation of the CHSH inequality that helps to pin down precisely how the correlations of a Bell inequality violating system are different to conventional classical correlations.

Consider a bipartite system where one part is sent to a LHS measuring device and the other to a RHS measuring device as sketched in Fig. 2.5. The LHS measuring device has a lever allowing it to measure either A or A'. The RHS measuring device can be set to B or B'. When a measurement is made the light under either "Yes" or "No" turns on. We are interested in the correlations between result combinations when measurements are made on the different settings.

Let the probabilities of different result combinations be written as $P(l, r|LR)$ where L and R are placeholders for the settings of the left and right measuring devices (i.e., L can take values A or A' and R values B and B') and l and r are placeholders for the results shown on the LHS and RHS measuring devices and as such can be either be"yes" or "no".

Bell inequalities define a correlation coefficient C as in Eq. (2.30) and then place an upper bound on possible values this coefficient can take if you assume "factorisability". **Factorisability** is the statement that the probability of l and r can be written as

$$p(l, r|LR) = \int P(l|L, \lambda)P(r|R, \lambda)P(\lambda)d\lambda \,. \tag{2.26}$$

What is the significance of the factorisability assumption ? If events x and y are uncorrelated then their joint distribution can be written as $P(x, y) = P(x)P(y)$. Similarly, the statement : $P(x, y|\alpha, \beta, \gamma) = P(x|\alpha, \beta, \gamma)P(y|\alpha, \beta, \gamma)$ says that the probabilities of x and y are uncorrelated once you take into account variables α, β and γ. Put another way, factors α, β and γ are sufficient to explain any correlations in the probabilities of x and y. For example, it seems reasonable to expect that the probability that a pub sells more than 100 ice creams in a day, $P(x)$, is correlated of the probability that the pub sells more than 1000 pints of cider, $P(y)$, but these correlations can be explained by taking into account all the various common factors such as outside temperature (α), day of the week (β), and the number of important sporting fixtures that day (γ). The parameter λ is introduced to incorporate all such common factors[4]

4. Note ; λ only includes factors from the events shared histories, it does not include explicit information about the results of either x or y. My example above would not be factorisable if a pub had a rule that every time 25 ice creams were sold they would toss a coin to decide whether to sell any more ciders that day.

and giving the original statement of factorisability, Eq. (2.26).

As such, the statement of "factorisability" used to set up the Bell inequality can be understood as follows. Given λ, the probability of the outcome of a particular measurement on the LHS given that A is measured, is uncorrelated to the probability of a particular result on the RHS, given that B is measured. That is, λ incorporates all effects from the system's shared history.

In terms of the experimental set up we are considering here λ represents all information concerning the initial state of the system and the experimental equipment before the system is divided and sent to the different measuring devices. As such, by denying that the joint probability distribution is factorisable we are denying that the correlations between the individual properties are explained by the local factors incorporated in λ. In this way, denying this form of correlation amounts to saying that the correlations are inexplicable in terms of local variables.

This idea can be made more precise by considering two necessary conditions for factorisability to hold.

1. **Setting Independence :** $P(l|L,B,\lambda) = P(l|L,B',\lambda)$

   The outcome on the LHS does not depend on what measurement is performed on the RHS and vice versa.

2. **Outcome Independence :** $P(l|A,R,r,\lambda) = P(l,|A,R,r',\lambda)$

   The outcome of LHS does not depend on the outcome of the outcome of the RHS, except in so far as them both depend on λ.

These two conditions lead to factorisability as follows. Given outcome independence, it makes sense to talk of individual probability distributions for l and r, and so we can say that

$$P(l,r|L,R,\lambda) = P(l|L,R,\lambda)P(r|L,R,\lambda) \tag{2.27}$$

Given setting independence we can further say that

$$P(l|L,R,\lambda) = P(l|L,\lambda) \tag{2.28}$$

and similarly for r. It thus follows that

$$P(l,r|L,R,\lambda) = P(l|L,\lambda)P(r|R,\lambda) \tag{2.29}$$

which leads directly to the factorizability condition Eq. (2.26). Thus, if a system is not factorizable then either outcome independence or setting independence (or both) does not hold.

In addition to factorizability two further implicit, but seemingly very reasonable assumptions, are required.

1. "Single outcome assumption" : On each run of the experiment only one result is obtained at each measuring device[5].
2. "No conspiracy assumption" : On each run on the experiment we only obtain results for one of four possible measurements ($A\&B$, $A'\&B$, $A\&B'$, $A'\&B'$). We find the probabilities required to calculate C by averaging out over many runs of the experiment. We need to assume that bias is not introduced by the measuring technique so that the samples used to calculate the probabilities are fair.

5. This may seem an odd assumption to explicitly state ; however, it does not hold under the many worlds interpretation of quantum mechanics.

Once you have these two definitions the rest of the derivation is basic probability and algebra. In what follows we present the original derivation by Bell which is slightly more general than that presented in the psychic section. Specifically, we will aim to bound

$$C := |\langle LR\rangle - \langle LR'\rangle| + |\langle LR\rangle + \langle L'R\rangle|\,. \tag{2.30}$$

Using the factorisability condition we have

$$\langle LR\rangle = \sum_{l,r=\pm1} lrP(l,r|L,R) \tag{2.31}$$

and similarly for the other terms in C.

Théorème 2.7.1. *Suppose that* ±1 *are the only allowed values for* l *and* r*. The "outcome independence", "setting independence", "single outcome" and "no conspiracy assumptions" above imply that*

$$C \leq 2$$

for all choices of parameters l, r, l', r'*.*

Démonstration.

For convenience let us implicitly define

$$\langle LR\rangle := E_{L,R}(l\cdot r) := \int E_{L,R}(l\cdot r|\lambda)P(\lambda)d\lambda = \sum_{l,r=\pm1} lrP(l,r|L,R)$$

where $E_{L,R}(l\cdot r)$ is the expectation value of the product $l\cdot r$ for a given choice of L and R. $E_{L,R}(l\cdot r|\lambda)$ represents the same quantity, conditioned on λ. Then we have

$$E_{L,R}(l,r|\lambda) = E_L(l|\lambda)E_R(r|\lambda) \quad \forall\lambda, L, R$$

from which

$$\begin{aligned}
C &= |\langle LR\rangle - \langle LR'\rangle| + |\langle LR\rangle + \langle L'R\rangle| \\
&\leq \int \big[|E_L(l|\lambda)|\cdot|E_R(r|\lambda) - E_{R'}(r|\lambda)| + |E_R(r|\lambda)|\cdot|E_L(l|\lambda) + E_{L'}(l|\lambda)|\big]P(\lambda)d\lambda \\
&\leq \int \big[|E_R(r|\lambda) - E_{R'}(r|\lambda)| + |E_L(l|\lambda) + E_{L'}(l|\lambda)|\big]P(\lambda)d\lambda
\end{aligned}$$

where the first inequality is taken from

$$\Big|\int f(x)dx\Big| \leq \int |f(x)|\,dx$$

and the second one

$$|E_\alpha(l|\lambda)| \leq 1$$

The proof of the theorem follows from

Lemme 2.7.2. *for* $x, y \in \mathbb{R}$ *and* $x, y \in [-1,1]$ *we have* $|x-y| + |x+y| \leq 2$

Démonstration.

$$\begin{aligned}
\big(|x-y| + |x+y|\big)^2 &= 2x^2 + 2y^2 + 2|x^2 - y^2| \\
&= \begin{cases} 4x^2 & x^2 > y^2 \\ 4y^2 & x^2 < y^2 \end{cases} \\
&\leq 4
\end{aligned}$$

□

Bell's non-locality theorem on its own does not tell us which of setting and outcome independence is violated quantum mechanics. However, violation of either of those criterions is sufficient to show that quantum mechanics is in some sense non-local. Bell's non locality theorem tells us either that the setting of the other measuring device, or the particular measurement made, affects the measurement on the other electron.

Note that there is nothing to prevent the measurement events at the two different devices from being spacelike, and so in terms of our current physical theories causally, separated. As such, either the information concerning the setting of the other measuring device, or result of the other measurement, is communicated at greater than the speed of light. However the former would violate the no signalling theorem. Hence we conclude that Quantum Mechanics violates outcome independence not parameter independence.

The correlation coefficient is constructed to apply to any physical theory which makes predictions for the probability of results in any experimental set up of the general structure outlined above. In particular, the derivation makes no direct appeal to either quantum mechanics or determinism. Experiments have subsequently confirmed that the CHSH-Bell inequality is violated by our world. This tells us that any fundamental physical theory for the world we live in (not just quantum mechanics but also any theory that makes accurate predictions about our world!) must have non-local features.

2.8 Contextuality

The final quantum property we will discuss in this chapter is contextuality. It is a less discussed quantum property but nicely completes the set discussed in this chapter so we will cover it in brief. The best example to get a quick sense of contextuality is the Peres-Mermin (PM) square introduced by Kochen and Specker.

Here we consider a set of 9 different binary measurements, each of which can give the outcomes ±1. Classically, we see this as being 9 properties of an object that we observe (+1) or do not observe (-1) in our system. We ask that observables in the same column or row form a context, or in other words, are jointly measurable.

$$\begin{bmatrix} A & B & C \\ a & b & c \\ \alpha & \beta & \gamma \end{bmatrix}$$

Let ABC denote the product of the values obtained from measuring A, B and C. Here, BC would be the measurement context of A. The observed properties can be probabilistic, so we define $\langle ABC \rangle = p(ABC = +1) - p(ABC = -1)$. We then consider (analogously to Bell inequalities) a correlation coefficient, this time of the form :

$$\langle PM \rangle = \langle ABC \rangle + \langle abc \rangle + \langle \alpha\beta\gamma \rangle + \langle Aa\alpha \rangle + \langle Bb\beta \rangle - \langle Cc\gamma \rangle \tag{2.32}$$

Classically we would expect measurements to be **noncontextual**. That is, we would expect the result of an observable to not depend on its context (the other measurements performed). If we assume our measurements are non-contextual then the maximum value the PM square can take is 4. In fact,

$$-4 \leq \langle PM \rangle \leq 4 \tag{2.33}$$

To see this note that the only way for the function f to have a value of 6 would be for all the products in the definition of f to be 1 except for the product cfi to be equal to -1. If the 5 first terms of the sum are all equals to 1, their product would also be equal to one, leading to :

$$a^2 b^2 d^2 e^2 g^2 h^2 cfi = 1,$$

implying that cfi is equal to 1. This proves that $f(M) \leq 4$. A similar argument can show that $f(M) \geq -4$.

However, by carefully picking our quantum observables, can show $\langle PM \rangle$ can *exceed* 4. Form the table of quantum observables as follows

$$\begin{bmatrix} A & B & C \\ a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} \quad \text{corresponding quantum example} \rightarrow \quad \begin{bmatrix} \sigma_z \otimes \mathbf{I} & \mathbf{I} \otimes \sigma_z & \sigma_z \otimes \sigma_z \\ \mathbf{I} \otimes \sigma_x & \sigma_x \otimes \mathbf{I} & \sigma_x \otimes \sigma_x \\ \sigma_z \otimes \sigma_x & \sigma_x \otimes \sigma_z & \sigma_y \otimes \sigma_y \end{bmatrix} \tag{2.34}$$

one can readily check that the columns and rows are made of commuting operators, and that the products of observables in the same contexts $\{A, B, C\}, \ldots$ are the identity except $Cc\gamma = -\mathbf{I}$. Thus we have $\langle PM \rangle = 6$ which violates Eq. (2.33). Note that this result is input state independent ! Any two qubit state (entangled or unentangled) is contextual. It follows that quantum mechanics is **contextual**. Broadly contextuality can be understood as stemming from the fact that observables in quantum mechanics do not commute. (Like Bell's inequality, violations of the PM bound have been experimentally verified.)